An overview of the 3D numerical method within UnTRIM²

Unstructured
n onlinear
Tidal
Residual
I nter-tidal
M² mud-flat Model

Vincenzo Casulli
Governing hydrostatic equations (3D)

\[
\begin{align*}
    u_t + uu_x + vu_y + wu_z &= -g \eta_x + (\mu u_x)_x + (\mu u_y)_y + (\mu u_z)_z \\
    v_t + uv_x + vv_y + wv_z &= -g \eta_y + (\mu v_x)_x + (\mu v_y)_y + (\mu v_z)_z \\
    u_x + v_y + w_z &= 0 \\
    (h + \eta)_t + \left( \int_{-h}^{\eta} u \, dz \right)_x + \left( \int_{-h}^{\eta} v \, dz \right)_y &= 0
\end{align*}
\]

- needs to be implicit for best (time) stability
- needs to be resolved for best (space) accuracy
- assumed to be sufficiently regular
Orthogonal unstructured grid

Volume: \( V_i(\eta_i^n) \)
Wet area: \( p_i(\eta_i^n) \)

3D velocity: \( u_{j,k}^n(y) \)
Layer thickness: \( \Delta z_{j,k}^n(y) \)
Vertical cross section

\[ \eta_j^n \]

\[ \Delta z_{j,k}^n (y) \quad u_{j,k}^n (y) \]

\[ H_j^n (y) = 0 \]
Semi-implicit momentum

\[
\begin{align*}
\frac{\partial}{\partial t} u_x + uu_x + vu_y + wu_z &= -g \eta_x + \left( \frac{\partial}{\partial x} (\mu u_x) \right)_x + \left( \frac{\partial}{\partial y} (\mu u_y) \right)_y + \left( \frac{\partial}{\partial z} (\mu u_z) \right)_z
\end{align*}
\]

\[
\Delta z^n_{j,k}(y)u^{n+1}_{j,k}(y) = G^n_{j,k}(y) - g \theta \Delta t \left( \eta^{n+1}_{r(j)} - \eta^{n+1}_{\ell(j)} \right) \Delta z^n_{j,k}(y)
\]

\[
+ \Delta t \left[ 
\begin{array}{c}
\mu^n_{j,k+\frac{1}{2}}(y) \frac{u^{n+1}_{j,k+1}(y) - u^{n+1}_{j,k}(y)}{\Delta z^n_{j,k+\frac{1}{2}}(y)} - \mu^n_{j,k-\frac{1}{2}}(y) \frac{u^{n+1}_{j,k}(y) - u^{n+1}_{j,k-1}(y)}{\Delta z^n_{j,k-\frac{1}{2}}(y)} \\
\gamma^n_{T,j}(y)[u^{n+1}_{a,j}(y) - u^{n+1}_{j,M}(y)] - \gamma^n_{B,j}(y)u^{n+1}_{j,m}(y)
\end{array}
\right]
\]

where

\[
k = m_j(y), m_j(y) + 1, \ldots, M^n_j
\]

\[
A^n_j(y)U^n_{j}(y) = G^n_j(y) - g \theta \frac{\Delta t}{\delta_j} \left( \eta^{n+1}_{r(j)} - \eta^{n+1}_{\ell(j)} \right) \Delta Z^n_j(y), \quad y \in \Gamma_j
\]
Semi-implicit free-surface

\[(h + \eta)_t + \left( \int_{-h}^{h} u dz \right)_x + \left( \int_{-h}^{h} v dz \right)_y = 0\]

\[V_i(\eta_i^{n+1}) = V_i(\eta_i^n) - \Delta t \sum_{j \in S_i} \left\{ \sigma_i,j \int_{\Gamma_j} \left[ \sigma_{m,j}(y) \left[ \theta U_{j,k}^{n+1}(y) + (1 - \theta) U_{j,k}^n(y) \right] \right] dy \right\} \]

\[V_i(\eta_i^{n+1}) = V_i(\eta_i^n) - \Delta t \sum_{j \in S_i} \sigma_{i,j} \int_{\Gamma_j} \left[ \sum_{k=m_j}^{M_j} \Delta Z_{j,k}(y) \left[ \theta U_{j,k}^{n+1}(y) + (1 - \theta) U_{j,k}^n(y) \right] \right] dy \]
Solution algorithm

Discrete wave equation (mildly nonlinear):

\[ V_i(\eta_i^{n+1}) - g \theta^2 \Delta t^2 \sum_{j \in S_i} \psi_j^n \frac{\eta_{n+1}^{(i,j)} - \eta_i^{n+1}}{\delta_j} = b_i^n, \quad i = 1, 2, \ldots, N_p \]

to be solved by Newton type iterations
Discrete wave equation

\[ V_i(\eta_i^{n+1}) = g \theta^2 \Delta t^2 \sum_{j \in S_i} \psi_j^n \frac{\eta_{\psi(i,j)}^{n+1} - \eta_i^{n+1}}{\delta_j} = b_i^n, \quad i = 1, 2, \ldots, N_p \]

\[ V_i(\eta_i^{n+1}) = \int_{\Omega_i} \max[0, h(x, y) + \eta_i^{n+1}] \, dx \, dy \geq 0 \quad \text{(nonlinear!)} \]

\[ \psi_j^n = \int_{\Gamma_j} [\Delta Z^T(y)A^{-1}(y)\Delta Z(y)]^n \, dy \geq 0 \]

\[ b_i^n = V_i(\eta_i^n) - \Delta t \sum_{j \in S_i, \Gamma_j} \sigma_{ij} \left\{ \Delta Z^T(y) \left[ \theta A^{-1}(y)G(y) + (1 - \theta)U(y) \right] \right\}^n_j \, dy \]

- Integrals replaced by summations over sub-polygons/edges
- Sub-grid can be arbitrarily chosen
Newton type iterations

\[ V_i(\eta_i^{n+1}) - g \theta^2 \Delta t^2 \sum_{j \in S_i} \psi_j^n \frac{\eta_{ij}^{n+1} - \eta_i^{n+1}}{\delta_j} = b_i^n, \quad i = 1, 2, \ldots, N_p \]

\[ \zeta = \eta^{n+1}, \quad V(\zeta) + T\zeta = b, \quad V_i(\zeta) = \int_{-\infty}^{\zeta} p_i(z)dz \]

\[ V'(\zeta) = P(\zeta), \quad p_i(z) \text{ surface wet area: nonnegative, non decreasing and bounded} \]

Newton iterations for \( V(\zeta) + T\zeta = b \): Guess \( \zeta^{(0)} = \eta^n \)

\[ \zeta^{(k+1)} = \zeta^{(k)} - [P(\zeta^{(k)}) + T]^{-1}[V(\zeta^{(k)}) + T\zeta^{(k)} - b], \quad k = 1, 2, \ldots \]
Horizontal velocity

\[ A_j^n(y)U_{j+1}^n(y) = G_j^n(y) - g \theta \frac{\Delta t}{\delta_j} (\eta_{r(j)}^{n+1} - \eta_{\ell(j)}^{n+1}) \Delta Z_j^n(y), \quad y \in \Gamma_j \]

\[ U_{j+1}^n(y) = \left[ A_j^n(y) \right]^{-1} \left[ G_j^n(y) - g \theta \frac{\Delta t}{\delta_j} (\eta_{r(j)}^{n+1} - \eta_{\ell(j)}^{n+1}) \Delta Z_j^n(y) \right], \quad y \in \Gamma_j \]

\[ a_{j,k}^n = \int_{\Gamma_j} \Delta z_{j,k}^n(y) dy \geq 0, \quad u_{j,k}^{n+1} = \frac{1}{a_{j,k}^n} \int_{\Gamma_j} \Delta z_{j,k}^n(y) u_{j,k}^{n+1}(y) dy \]
Finite volume for vertical velocity

\[ u_x + v_y + w_z = 0 \]

\[ p_{i,k+\frac{1}{2}} w_{i,k+\frac{1}{2}}^{n+1} - p_{i,k-\frac{1}{2}} w_{i,k-\frac{1}{2}}^{n+1} + \sum_{j \in S_i} \left[ \sigma_{i,j} \int_{\Gamma_j} \Delta z_{j,k}^{n+1} (y) u_{j,k}^{n+1} (y) dy \right] = 0 \]

\[ w_{i,m-\frac{1}{2}}^{n+1} = 0; \]

\[ w_{i,k+\frac{1}{2}}^{n+1} = \frac{1}{p_{i,k+\frac{1}{2}}} \left\{ p_{i,k-\frac{1}{2}} w_{i,k-\frac{1}{2}}^{n+1} - \sum_{j \in S_i} \left[ \sigma_{i,j} \int_{\Gamma_j} \Delta z_{j,k}^{n+1} (y) u_{j,k}^{n+1} (y) dy \right] \right\} \]

\[ k = m_i, m_i + 1, \ldots, M_i^n - 1 \]
\[ u = \frac{1}{2[1 - A \cos \omega t]} \left\{ \omega x A \sin \omega t - f y \left[ \sqrt{1 - A^2} + A \cos \omega t - 1 \right] \right\} \]

\[ v = \frac{1}{2[1 - A \cos \omega t]} \left\{ \omega y A \sin \omega t + f x \left[ \sqrt{1 - A^2} + A \cos \omega t - 1 \right] \right\} \]

\[ w = \frac{\omega A \sin \omega t}{1 - A \cos \omega t} \left( 2h_o \frac{x^2 + y^2}{L^2} - h_o - z \right) \]

\[ \eta = h_o \left\{ \frac{\sqrt{1 - A^2}}{1 - A \cos \omega t} - 1 - \frac{x^2 + y^2}{L^2} \left[ \frac{1 - A^2}{(1 - A \cos \omega t)^2} - 1 \right] \right\} \]

\[ A_{\text{wet}} = \pi L^2 \frac{1 - A \cos(\omega t)}{\sqrt{1 - A^2}} \]
Exact shoreline
Despite coarse Cartesian grid…... accurate boundary fitting!
Comparisons with the analytical solution

<table>
<thead>
<tr>
<th>Radial velocity</th>
<th>Tangential velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(due to gravity)</td>
<td>(due to Coriolis)</td>
</tr>
<tr>
<td><img src="image" alt="Radial velocity graph" /></td>
<td><img src="image" alt="Tangential velocity graph" /></td>
</tr>
</tbody>
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<table>
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<tr>
<th>Vertical velocity</th>
<th>Wet area</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Vertical velocity graph" /></td>
<td><img src="image" alt="Wet area graph" /></td>
</tr>
</tbody>
</table>

Costly grid refinement for boundary fitting and front tracking can be avoided.
Venice Lagoon (high resolution 25m)

\[ N_p = 671,030 \]
\[ N_s = 1,361,331 \]
\[ N_k = 50 \]
\[ \Delta t = 300s \]
Coarse, non-linear Cartesian grid (300m)

\[ N_p = 5,627 \]
\[ N_s = 11,959 \]
\[ N_k = 50 \]
\[ \Delta t = 300s \]

\[ n_{s,j}^{SG} = 12 \]
\[ n_{p,i}^{SG} = 144 \]

Observation

Speedup: 408
3D axially symmetric flows
(reduced space and time scale)

\[ u_t + uu_x + wu_z = -p_x + \frac{v}{z}(zu)_z \] \text{axial momentum}

\[ (zu)_x + (zw)_z = 0 \] \text{incompressibility}

\[ A_t + 2\pi \left( \int_0^R zdz \right)_x = 0 \] \text{free-surface}

\[ A = \pi R^2, \quad p = p_{ext} + \beta(R - R_0) \] \text{Laplace law (equation of state)}

Unknowns: \( u(x, z, t), w(x, z, t), p(x, t) \)
Semi-implicit discretization

Discrete wave equation:

\[ V_i(p_i^{n+1}) - g \theta^2 \Delta t^2 \sum_{j \in S_i} \psi_j^n \frac{p_i^{n+1} - p_j^{n+1}}{\Delta x_j} = b_i^n, \quad i = 1, 2, \ldots, N_p \]
Blood flow in a complex arterial system

\[ N_k = 30 \]
\[ N_p = 1,448 \]
\[ N_s = 1,447 \]
\[ \Delta x_j = 0.005m \]
\[ \Delta z = 0.001m \]
\[ \Delta t = 0.01s \]
Conclusions

- Any polygon is allowed to be wet, partially wet or dry;
- Even coarse grids, naturally resolve detailed water fronts;
- Volume is a nonnegative nonlinear function of $\eta$;
- Detailed boundary fitting obtained at sub-grid level;
- Precise volume and mass balance at sub-grid level;
- \textbf{3D} $\rightarrow$ \textbf{2D} when/where one vertical layer is specified;
  - Reduced problem size;
  - Small number of iterations;
  - Drastically reduced the overall computational effort;
- Applies to axially symmetric blood flow in compliant arteries.
Thank you